Available at www.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT
Journal of Approximation Theory 123 (2003) 117–124
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

A best constant for bivariate Bernstein and Szász-Mirakyan operators ☆

Jesús De La Cal,* Javier Cárcamo, and Ana M. Valle

Departamento de Matemática Aplicada y Estadística, e Investigación Operativa, Facultad de Ciencias, Universidad del País Vasco. Apartado 644, 48080 Bilbao, Spain

Received 9 August 2002; accepted in revised form 23 April 2003

Communicated by Dany Leviatan

Abstract

For classical Bernstein operators over the unit square, we obtain the best uniform constant in preservation of the usual l_∞-modulus of continuity, at the same time we show that it coincides with the corresponding best uniform constant for bivariate Szász operators. The result validates a conjecture stated in a previous paper. The proof involves both probabilistic and analytic arguments, as well as numerical computation of some specific values.

Keywords: Best constants; Modulus of continuity; Bernstein operators; Szász–Mirakyan operators; Bivariate operators; Binomial distribution; Poisson distribution

1. Introduction and main result

For $n, k = 1, 2, \ldots$ let $B_n^{\langle k \rangle} := B_n \otimes \cdots \otimes B_n$ be the tensor product of k copies of the classical Bernstein operator over the interval $[0, 1]$ given by

$$B_n f(x) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k},$$

☆ Research supported by the Spanish Government PB98-1577-C02-02 and BFM2002-04163-C02-02 Grants, and by FEDER.

*Corresponding author. Fax: 34-94-464-8500.

E-mail addresses: mepcaagj@lg.ehu.es (J. De La Cal), mebcaurj@lg.ehu.es (J. Cárcamo), mepvamaa@lg.ehu.es (A.M. Valle).

doi:10.1016/S0021-9045(03)00086-8

and denote by $C_n^{\langle k \rangle}(\delta)$ the best constant in preservation of the usual modulus of continuity for the l_∞-norm in $\mathbb{R}^k$, that is

$$C_n^{\langle k \rangle}(\delta) := \sup_{f \in \mathscr{F}_k} \frac{\omega(B_n^{\langle k \rangle} f; \delta)}{\omega(f; \delta)}, \quad 0 < \delta \leqslant 1,$$

where $\mathscr{F}_k$ is the set of all real non-constant bounded functions on $[0,1]^k$, and $\omega(f;\cdot)$ stands for the above-mentioned modulus of continuity, i.e.,

$$\omega(f;\delta) := \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in [0,1]^k, \ ||\mathbf{x} - \mathbf{y}||_\infty \leqslant \delta\}, \quad \delta \geqslant 0.$$

On the other hand, for $t > 0$ and $k = 1, 2, \ldots$, let $S_t^{\langle k \rangle} := S_t \otimes \cdots \otimes S_t$ be the tensor product of k copies of the Szász–Mirakyan operator S_t over the interval $[0, \infty)$ given by

$$S_t g(x) := \sum_{k=0}^{\infty} g(k/t) e^{-tx} \frac{(tx)^k}{k!},$$

and denote by $D_t^{\langle k \rangle}(\delta)$ the corresponding best constant for such an operator, i.e.,

$$D_t^{\langle k \rangle}(\delta) := \sup_{g \in \mathscr{G}_k} \frac{\omega(S_t^{\langle k \rangle} g; \delta)}{\omega(g; \delta)}, \quad \delta > 0,$$

where $\mathscr{G}_k$ is the set of all real non-constant functions g on $[0, \infty)^k$ such that $\omega(g; 1) < \infty$ (or, equivalently, $\omega(g; \delta) < \infty$ for all $\delta > 0$).

Many facts about $C_n^{\langle k \rangle}(\delta)$ and $D_t^{\langle k \rangle}(\delta)$ are known in the literature. In [4], the reader can find explicit probabilistic formulae for such constants (to be used in the next section) which generalize the one-dimensional formulae given in [1]. From the formula for $D_t^{\langle k \rangle}(\delta)$, it follows that

$$D_t^{\langle k \rangle}(\delta) = D_1^{\langle k \rangle}(t\delta), \quad t, \delta > 0,$$

implying that the best uniform constant $\sup_{\delta > 0} D_t^{\langle k \rangle}(\delta)$ only depends upon the dimension k. The facts

$$\sup_{0 < \delta \leqslant 1} C_n^{\langle 1 \rangle}(\delta) = 2 \quad (n \geqslant 1) \quad \text{and} \quad \sup_{\delta > 0} D_t^{\langle 1 \rangle}(\delta) = 2 - e^{-1} \quad (t > 0)$$

were respectively established in [2,6]. It was shown in [5] that, for $k \geqslant 3$,

$$\sup_{0 < \delta \leqslant 1} C_n^{\langle k \rangle}(\delta) = k = \sup_{\delta > 0} D_1^{\langle k \rangle}(\delta), \quad n \geqslant 1,$$

that is, the best uniform constants coincide with the dimension, and (as in the one-dimensional case) the one for $B_n^{\langle k \rangle}$ does not depend upon the parameter n, while, in the case $k = 2$, the value of $\sup_{0 < \delta \leqslant 1} C_n^{\langle 2 \rangle}(\delta)$ *depends* upon n, and both the values of $\sup_{n \geqslant 1} \sup_{0 < \delta \leqslant 1} C_n^{\langle 2 \rangle}(\delta)$ and $\sup_{\delta > 0} D_1^{\langle 2 \rangle}(\delta)$ lie in the interval $[2, 5/2]$. As for the exact value of these quantities, on the basis of certain computational evidence, it was conjectured the following.

Theorem. *We have*

$$\sup_{\delta>0} D_1^{\langle 2\rangle}(\delta) = \sup_{n\geqslant 1} \sup_{0<\delta\leqslant 1} C_n^{\langle 2\rangle}(\delta)$$
$$= 1 - e^{-2} + \sum_{j=0}^{\infty}\left[1 - e^{-2}\left(\sum_{i=0}^{j}\frac{1}{i!}\right)^2\right] = 2.3884423\ldots \quad . \tag{1}$$

In the present paper, we give a theoretical proof of this result.

2. Auxiliary results

In this section, we introduce some notations, restate the preceding theorem in a more convenient form for our purposes, and collect some necessary auxiliary results.

We set, for $n\geqslant 1$ and $0<x\leqslant n$,

$$C_n(x) := C_n^{\langle 2\rangle}(x/n).$$

We recall that, according to the formulae in [3], we have

$$C_n(x) = E\left\lceil\frac{\eta_n(x)}{x}\right\rceil,$$

where E denotes mathematical expectation, $\lceil\cdot\rceil$ is the ceiling function, and $\eta_n(x) := \eta_n'(x)\vee\eta_n''(x)$ is the maximum of two independent integer-valued random variables $\eta_n'(x)$ and $\eta_n''(x)$ having the same binomial distribution given by

$$P(\eta_n'(x)=k) = p_{n,k}(x) := \begin{cases} \binom{n}{k}\left(\frac{x}{n}\right)^k\left(1-\frac{x}{n}\right)^{n-k} & k=0,1,\ldots,n, \\ 0 & \text{otherwise.}\end{cases}$$

We also denote by

$$C_n^*(x) := P(\eta_n(x)>0) + \frac{E\eta_n(x)}{x} = 1 - p_{n,0}^2(x) + \frac{E\eta_n(x)}{x},$$

and recall that

$$E\eta_n(x) = \sum_{k=1}^{\infty} kP(\eta_n(x)=k) = \sum_{k=1}^{\infty} P(\eta_n(x)\geqslant k)$$
$$= \sum_{k=0}^{n-1}\left[1-\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2\right].$$

Similarly, we set, for $x>0$,

$$D(x) := D_1^{\langle 2\rangle}(x) = E\left\lceil\frac{\xi(x)}{x}\right\rceil,$$

and

$$D^*(x) := P(\xi(x) > 0) + \frac{E\xi(x)}{x}$$
$$= 1 - \pi_0^2(x) + \frac{1}{x}\sum_{k=0}^{\infty}\left[1 - \left(\sum_{j=0}^{k}\pi_j(x)\right)^2\right],$$

where $\xi(x) := \xi'(x) \vee \xi''(x)$ is the maximum of two independent random variables $\xi'(x)$ and $\xi''(x)$ having the same Poisson distribution of parameter x, i.e.,

$$P(\xi'(x) = k) = \pi_k(x) := e^{-x}\frac{x^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

With the preceding notations, it is clear that (1) can be rewritten as follows:

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) = \sup_{x > 0} D(x) = D^*(1). \tag{2}$$

The point is that the functions $C_n(\cdot)$ and $D(\cdot)$ are quite irregular and hardly tractable, but $C_n^*(\cdot)$ and $D^*(\cdot)$ are fairly smooth. The following lemmas collect the necessary facts for the proof of (2) given in the next section.

Lemma 1. *We have*

(a)

$$C_n(x) \leqslant C_n^*(x), \quad n \geqslant 1, \quad 0 < x \leqslant n.$$

(b)

$$D(x) \leqslant D^*(x), \quad x > 0.$$

Proof. Both inequalities are nothing but particular cases of the inequality

$$E\lceil U \rceil \leqslant P(U > 0) + EU,$$

which holds true for every nonnegative random variable U. $\square$

Lemma 2. *We have*

$$C_n^*(x) \leqslant D^*(x), \quad n \geqslant 1, \ 0 < x \leqslant n.$$

Proof. Fix $n \geqslant 1$ and $x \in (0, n]$, and denote by

$$a_{n,k} := P(\eta_n'(x) \leqslant k) = \sum_{j=0}^{k} p_{n,j}(x),$$

$$b_k := P(\xi'(x) \leqslant k) = \sum_{j=0}^{k} \pi_j(x),$$

for $k = 0, 1, 2, \ldots$. From some results of Anderson and Samuels [3, Corollary 2.1], there is an integer $r \geqslant 1$ such that

$$a_{n,k} \leqslant b_k, \quad 0 \leqslant k \leqslant r-1,$$

$$a_{n,k} \geqslant b_k, \quad k \geqslant r. \tag{3}$$

We have

$$\begin{aligned} E\xi(x) - E\eta_n(x) &= \sum_{k=0}^{\infty} (1 - b_k^2) - \sum_{k=0}^{\infty} (1 - a_{n,k}^2) \\ &= \sum_{k=0}^{\infty} (a_{n,k}^2 - b_k^2) \\ &= a_{n,0}^2 - b_0^2 + \sum_{k=1}^{\infty} (a_{n,k} + b_k)(a_{n,k} - b_k) \\ &\geqslant a_{n,0}^2 - b_0^2 + (a_{n,r} + b_r) \sum_{k=1}^{\infty} (a_{n,k} - b_k), \end{aligned}$$

the inequality by (3) and the fact that the sequence $\{a_{n,k} + b_k : k \geqslant 0\}$ is nondecreasing. Since

$$\begin{aligned} \sum_{k=1}^{\infty} (a_{n,k} - b_k) &= \left[\sum_{k=0}^{\infty} (1 - b_k) - \sum_{k=0}^{\infty} (1 - a_{n,k}) \right] - (a_{n,0} - b_0) \\ &= [E\xi'(x) - E\eta_n{}'(x)] - (a_{n,0} - b_0) \\ &= -(a_{n,0} - b_0) \end{aligned}$$

(the last equality because $E\xi'(x) = x = E\eta_n{}'(x)$), and

$$a_{n,r} + b_r \geqslant a_{n,1} + b_1 \geqslant (1+x)(a_{n,0} + b_0),$$

we finally obtain that

$$E\xi(x) - E\eta_n(x) \geqslant x(b_0^2 - a_{n,0}^2),$$

which is another way to express the conclusion. □

Lemma 3. *We have*:

(a)

$$x \frac{d}{dx} E\eta_n(x) = E\eta_n(x) - \sum_{k=1}^{n} k p_{n,k}^2(x), \quad n \geqslant 1, \ x \in (0, n].$$

(b)

$$x \frac{d}{dx} E\xi(x) = E\xi(x) - \sum_{k=1}^{\infty} k \pi_k^2(x), \quad x > 0.$$

Proof. Let $n \geqslant 1$ be fixed. It is immediate that we have, for $x \in (0, n]$ and $k \geqslant 0$,

$$x \frac{d}{dx} p_{n,k}(x) = k p_{n,k}(x) - (k+1) p_{n,k+1}(x),$$

implying that

$$x\frac{d}{dx}\sum_{j=0}^{k} p_{n,j}(x) = -(k+1)p_{n,k+1}(x),$$

and, therefore,

$$\begin{aligned} x\frac{d}{dx}\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2 &= -(k+1)2P(\eta_n{}'(x)\leqslant k)P(\eta_n''(x)=k+1) \\ &= -(k+1)[P(\eta_n{}'(x)\leqslant k)P(\eta_n''(x)=k+1) \\ &\quad + P(\eta_n''(x)\leqslant k)P(\eta_n{}'(x)=k+1)] \\ &= -(k+1)[P(\eta_n(x)=k+1)-p_{n,k+1}^2(x)]. \end{aligned}$$

We conclude that

$$\begin{aligned} x\frac{d}{dx}E\eta_n(x) &= x\frac{d}{dx}\sum_{k=0}^{n-1}\left[1-\left(\sum_{j=0}^{k} p_{n,j}(x)\right)^2\right] \\ &= \sum_{k=1}^{\infty} kP(\eta_n(x)=k) - \sum_{k=1}^{n} kp_{n,k}^2(x) \\ &= E\eta_n(x) - \sum_{k=1}^{n} kp_{n,k}^2(x), \end{aligned}$$

showing part (a). The proof of (b) is achieved in the same way, by starting from the fact that we have

$$x\frac{d}{dx}\pi_k(x) = k\pi_k(x) - (k+1)\pi_{k+1}(x),$$

for all $x>0$ and $k\geqslant 0$. □

Lemma 4. (a) *For each* $n\geqslant 1$, *the function* $x^{-1}E\eta_n(x)$ *is decreasing in* $(0,n]$.
(b) *The function* $x^{-1}E\xi(x)$ *is decreasing in* $(0,\infty)$.
(c) *The function* $D^*(\cdot)$ *is increasing in* $(0,1]$ *and decreasing in* $[3/2,\infty)$.

Proof. From the preceding lemma, we have

$$\frac{d}{dx}\frac{E\eta_n(x)}{x} = -\frac{1}{x^2}\sum_{k=1}^{n} kp_{n,k}^2(x) < 0, \quad n\geqslant 1,\ x\in(0,n],$$

and

$$\frac{d}{dx}\frac{E\xi(x)}{x} = -\frac{1}{x^2}\sum_{k=1}^{\infty} k\pi_k^2(x) = -\sum_{k=0}^{\infty}\frac{1}{k+1}\pi_k^2(x) < 0, \quad x>0, \tag{4}$$

showing parts (a) and (b). From (4), we also have, for $x>0$,

$$\begin{aligned}\frac{d}{dx}D^*(x) &= \frac{d}{dx}\left[1-\pi_0^2(x)+\frac{E\xi(x)}{x}\right]\\ &= 2\pi_0^2(x)-\sum_{k=0}^{\infty}\frac{1}{k+1}\pi_k^2(x)\\ &= \pi_0^2(x)-\sum_{k=1}^{\infty}\frac{1}{k+1}\pi_k^2(x)\\ &= e^{-2x}\left[1-\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2\right].\end{aligned}$$

If $x\in(0,1]$, we have

$$\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2 \leqslant \sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{1}{k!}\right)^2 < \frac{1}{2}(e-1)<1,$$

implying that $\frac{d}{dx}D^*(x)>0$, while, for $x\geqslant 3/2$,

$$\sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{x^k}{k!}\right)^2 \geqslant \sum_{k=1}^{\infty}\frac{1}{k+1}\left(\frac{(3/2)^k}{k!}\right)^2 > \frac{1}{2}\left(\frac{3}{2}\right)^2>1,$$

which implies that $\frac{d}{dx}D^*(x)<0$. This shows part (c), and completes the proof of the lemma. □

Lemma 5. *We have*:

(a) $\lim_{x\uparrow 1} C_n(x) = C_n^*(1)$, $n\geqslant 1$.
(b) $\lim_{n\to\infty} C_n^*(1) = D^*(1)$.
(c) $\lim_{x\uparrow 1} D(x) = D^*(1)$.
(d) $E\xi(1) = D^*(1) - 1 + e^{-2} = 1.52377761\ldots$.
(e) $D^*(1.55) = 2.38835554\ldots$.

Proof. We have, for $n\geqslant 1$,

$$\begin{aligned}\lim_{x\uparrow 1} C_n(x) &= \lim_{x\uparrow 1}\sum_{k=1}^{n}\left\lceil\frac{k}{x}\right\rceil P(\eta_n(x)=k) = \sum_{k=1}^{n}(k+1)P(\eta_n(1)=k)\\ &= P(\eta_n(1)>0) + E\eta_n(1) = C_n^*(1),\end{aligned}$$

showing part (a). Part (c) is shown in the same way, and we omit the details. Part (b) readily follows from the fact that

$$\lim_{n\to\infty} p_{n,k}(1) = \pi_k(1), \quad k=0,1,2,\ldots$$

(i.e., the Poisson approximation to the binomial distribution). Finally, parts (d) and (e) merely are numerical computations. □

3. Proof of the theorem

Recall the numerical value of $D^*(1)$ appearing in (1). We have, by Lemmas 1, 2, 4(c) and 5(e),

$$D(x) \leqslant D^*(x) \leqslant D^*(1), \quad 0 < x \leqslant 1,$$

$$C_n(x) \leqslant C_n^*(x) \leqslant D^*(x) \leqslant D^*(1), \quad 0 < x \leqslant 1, \ n \geqslant 1,$$

$$D(x) \leqslant D^*(x) \leqslant D^*(1.55) < D^*(1), \quad x \geqslant 1.55,$$

$$C_n(x) \leqslant C_n^*(x) \leqslant D^*(x) \leqslant D^*(1.55) < D^*(1), \quad n \geqslant 2, \ 1.55 \leqslant x \leqslant n.$$

Let $1 < x < 1.55$. Using the fact that the random variable $\xi(x)$ is integer-valued, and Lemmas 4(b) and 5(d), we obtain

$$D(x) \leqslant E\lceil \xi(x) \rceil = E\xi(x) \leqslant xE\xi(1)$$
$$\leqslant 1.55E\xi(1) = 2.361855\ldots < D^*(1),$$

and, analogously, by Lemmas 2 and 4(a),

$$C_n(x) \leqslant E\lceil \eta_n(x) \rceil = E\eta_n(x) \leqslant xE\eta_n(1) \leqslant x\,E\xi(1) < D^*(1), \quad n \geqslant 2.$$

From all the above, we conclude that

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) \leqslant D^*(1) \quad \text{and} \quad \sup_{x > 0} D(x) \leqslant D^*(1). \tag{5}$$

Finally, we have, by Lemma 5(c)

$$\sup_{x > 0} D(x) \geqslant \lim_{x \uparrow 1} D(x) = D^*(1),$$

and, by Lemma 5(a,b),

$$\sup_{n \geqslant 1} \sup_{0 < x \leqslant n} C_n(x) \geqslant \lim_{n \to \infty} \lim_{x \uparrow 1} C_n(x) = D^*(1),$$

showing that the inequalities in (5) actually are equalities, and finishing the proof of (2).

References

[1] J.A. Adell, A. Pérez-Palomares, Best constants in preservation inequalities concerning the first modulus and Lipschitz classes for Bernstein-type operators, J. Approx. Theory 93 (1998) 128–139.

[2] G.A. Anastassiou, C. Cottin, H.H. Gonska, Global smoothness of approximating functions, Analysis 11 (1991) 43–57.

[3] T.W. Anderson, S.M. Samuels, Some inequalities among binomial and Poisson probabilities, Proceedings of the Fifth Berkeley Symposium on Mathematics Statistics and Probability, Vol. 1, University of California Press, Berkeley, CA, 1967, pp. 1–12.

[4] J. De La Cal, A.M. Valle, Global smoothness preservation by multivariate Bernstein-type operators, in: G.A. Anastassiou (Ed.), Handbook on Analytic Computational Methods in Applied Mathematics, CRC Press, LLC, Boca Raton, 2000, pp. 667–707.

[5] J. De La Cal, A.M. Valle, Best constants for tensor products of Bernstein, Szász and Baskakov operators, Bull. Austral. Math. Soc. 62 (2000) 211–220.

[6] A. Pérez-Palomares, Global smoothness preservation properties for generalized Szász-Kantorovich operators, preprint.